



Graph Theory.

2. Vertex Descriptors and Graph Coloring

Lorentz JÄNTSCHI

Technical University Cluj-Napoca

<http://lori.academicdirect.ro>

Abstract

This original work presents the construction of a set of ten sequence matrices and their applications for ordering vertices in graphs. For every sequence matrix three ordering criteria are applied: lexicographic ordering, based on strings of numbers, corresponding to every vertex, extracted as rows from sequence matrices; ordering by the sum of path lengths from a given vertex; and ordering by the sum of paths, starting from a given vertex.

We also examine a graph that has different orderings for the above criteria. We then proceed to demonstrate that every criterion induced its own partition of graph vertex.

We propose the following theoretical result: both LAVS and LVDS criteria generate identical partitioning of vertices in any graph.

Finally, a coloring of graph vertices according to introduced ordering criteria was proposed.

Keywords

Graph theory, Vertex descriptors, Matrix based descriptors, Invariants, Graph coloring, Graph partitioning.

1. Introduction

The problem of vertex partitioning in graphs can be solved in different ways [1,2]. The most frequently used discriminating criteria are well-known square matrices that collect contributions of pairs of vertices from the graph. The adjacency matrix is the simplest matrix of this type. Some authors use characteristic polynomial and its eigenvalue to characterize a given graph [3]. The sequence of eigenvalues forms the graph spectrum [4]. A basic result in spectral geometry is the theorem of Cheng [5].

Let us consider a sequence matrix. It is of dimension $n \times (n-1)$, where n is the number of vertices. Such a matrix can be used to compare different graphs (interstructural similarity), when analyzing the columns of sequence matrices, the sums by columns or global sums (generic indices) [6]. It is also possible to compare the vertices of a graph (intrastructural similarity), when analyzing the rows of sequence matrices and the sums by rows [7,8]. With this observation, a sequence matrix is an elegant method to investigate graphs [9]. Note that the ordering of graph vertices by using sequence matrices is of a partially ordered type [10,11].

Prior, Dobrynin reported the degeneracy of some sequence matrices [12,13]. Decomposition of graphs into congruent factors has interesting chemical implications [14,15]. Dobrynin [16] and Diudea [17,18] reported distance degree sequences (DPS in our notation) and their chemical applications. The path sequence matrix (APS in our notation) is calculated in order to study molecular similarity [19].

2. Theoretical Considerations

Some definitions are required. Let $G = (V, E)$ be an un-oriented graph. We note the set of all path in G with $P(G)$. The distance from i to j in G is $d(G)_{i,j}$ and detour is $\delta(G)_{i,j}$. For $p \in P(G)$, $l(p)$ denotes the length of path p .

$t = (v_i)_{1 \leq i \leq n}$ is a *terminal path* if:

(1) $t \in P(G)$ and

(2) $\forall v \in V$ s. th. $(v_n, v) \in E$ then $t \cup (v_n, v) \notin P(G)$.

The set of terminal paths starting with i in G is $T(G)_i = \{t = (v_i)_{1 \leq i \leq n}$ s. th. $v_1 = i$ and t terminal path}. The set of terminal paths in G is $T(G) = (T(G)_v)_{v \in V}$.

The set of distance paths from i to j in G is $DP(G)_{i,j} = \{p=(x_i)_{1 \leq i \leq n} \text{ s. th. } p \in P(G), x_1=i, x_n=j \text{ \& } l(p)=d(G)_{i,j}\}$. The set of distance path in G is $DP(G) = \{p = (x_i)_{1 \leq i \leq n} \text{ s. th. } p \in P(G) \text{ and } l(p) = d(G)_{x_1, x_n}\}$.

The set of detour paths from i to j in G is $\Delta P(G)_{i,j} = \{p=(x_i)_{1 \leq i \leq n} \text{ s. th. } p \in P(G), x_1=i, x_n=j \text{ \& } l(p)=\delta(G)_{i,j}\}$. The set of detour path in G is $\Delta P(G) = \{p = (x_i)_{1 \leq i \leq n} \text{ s. th. } p \in P(G) \text{ and } l(p) = \delta(G)_{x_1, x_n}\}$.

The set of distance terminal paths from i to j in G is $DT(G)_{i,j} = \{t=(x_i)_{1 \leq i \leq n} \text{ s. th. } t \in T(G), x_1=i, x_n=j \text{ \& } l(t)=d(G)_{i,j}\}$.

The set of distance terminal path in G is $DT(G) = \{t = (x_i)_{1 \leq i \leq n} \text{ s. th. } t \in T(G) \text{ and } l(t) = d(G)_{x_1, x_n}\}$.

The set of detour terminal paths from i to j in G is $\Delta T(G)_{i,j} = \{t=(x_i)_{1 \leq i \leq n} \text{ s. th. } t \in T(G), x_1=i, x_n=j \text{ \& } l(t)=\delta(G)_{i,j}\}$.

The set of detour terminal path in G is $\Delta T(G) = \{t = (x_i)_{1 \leq i \leq n} \text{ s. th. } t \in T(G) \text{ and } l(t) = \delta(G)_{x_1, x_n}\}$.

Note that if $G = (V, E)$ is a connected graph, then $\forall v \in V, \forall e \in E \exists t \in T(G)_v$ s. th. $e \subseteq t$. Thus, any connected graph can be reconstructed from the terminal path list starting from an arbitrary vertex. Also, $\forall i, j \in V, T(G)_j$ can be constructed from $T(G)_i$. Consequently, the sets of terminal paths $T(G)_v, v \in V$ are equivalent by construction.

In order to generate sequence matrices, we compute terminal paths. Let $G = (V, E), V = \{1, \dots, 8\}, E = \{(1, 2), (1, 3), (2, 4), (3, 5), (3, 6), (4, 5), (4, 7), (6, 7), (7,8)\}$ shown in figure 1:

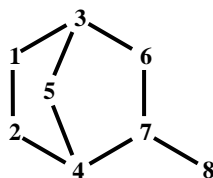


Fig. 1. Graph G (see above)

We have devised a 16-bit windows original computer program to list all terminal paths. For the graph in figure 1, we have obtained the following set of terminal paths for vertex 1:

								T(G) ₁
1	2	4	5	3	6	7	8	
1	2	4	7	8				
1	2	4	7	6	3	5		
1	3	6	7	8				
1	3	6	7	4	2			
1	3	6	7	4	5			
1	3	5	4	7	6			
1	3	5	4	7	8			
1	3	5	4	2				

Fig. 2. Terminal paths of graph G from fig. 1 starting from vertex 1

3. The Construction of Sequence Matrices of Numbering

We extend the concept used for the distance degree sequence matrix and layer path (path degree sequence) matrix to construct a set of 10 sequence matrices. A sequence matrix is a matrix that collects vertex contribution in each row and elongation of paths in each column. The elements of the matrix cumulate a value of vertex property for a given elongation. Sequence matrices are of dimension $n \times (n-1)$ and are given by following definitions:

SMN (sequence matrix of numbering) of paths (P) which starts with i and has length j is $APS(G)_{i,j} = |\{p \in P(G), \text{ s. th. } p_1 = i, l(p) = j\}|$, where p_1 is the first vertex from path p and operator $|\cdot|$ is the cardinal operator.

SMN of distance P that starts with i and has length j is $DPS(G)_{i,j} = |\{p \in APS(G)_{i,j}, \text{ s. th. } p \in DP(G)\}|$.

SMN of detour P that starts with i and has length j is $\Delta PS(G)_{i,j} = |\{p \in APS(G)_{i,j}, \text{ s. th. } p \in \Delta P(G)\}|$.

SMN of terminal paths (T) which starts with i and has length j is $ATS(G)_{i,j} = |\{t \in T(G), \text{ s. th. } t_1 = i, l(t) = j\}|$.

SMN of distance T that starts with i and has length j is $DTS(G)_{i,j} = |\{p \in ATS(G)_{i,j}, \text{ s. th. } p \in DT(G)\}|$.

SMN of detour T that starts with i and has length j is $\Delta TS(G)_{i,j} = |\{p \in AT(S)(G)_{i,j}, \text{ s. th. } p \in \Delta T(G)\}|$.

SMN of the vertex situated at the end of paths that starts with i and has j edges is $AVS(G)_{i,j} = |\{v \in V, \text{ s. th. } \exists p \in P(G), p_1 = i, p_{j+1} = v, l(p) = j\}|$.

SMN of the vertex situated at the end of distance paths that starts with i and has j edges is $DVS(G)_{i,j} = |\{v \in V, \text{ s. th. } \exists p \in DP(G), p_1 = i, p_{j+1} = v, l(p)=j\}|$.

SMN of the vertex situated at the end of detour paths that starts with i and has j edges is $\Delta VS(G)_{i,j} = |\{v \in V, \text{ s. th. } \exists p \in \Delta P(G), p_1 = i, p_{j+1} = v, l(p)=j\}|$.

SMN of the cycles that contain vertex i and has j edges is $ACS(G)_{i,j} = |\{p \in P(G), \text{ s. th. } l(p) = j-1 \text{ and } (p_j, p_1) \in E\}|$.

If an lexicographically ordering are applied to the sequence matrices of numbering (SMN), through permutation of one line with another, and the new matrix obtained can be used to make interstructural comparisons (with any other graph) because a lexicographically ordered matrix is independent with respect to labeling.

When two vertices are compared, lexicographic ordering is also useful from the point of view of the selected criterion.

The results of lexicographic ordering can be used to index the graph, by permuting the old indexing.

The computer program used in lexicographic ordering (*quicksort* method), gives permutation of partial ordering (σ in what follows).

Ten sequence matrices, ten lexicographically associated matrices and ten permutations σ are computed.

Two vertex indices in every case (from ten cases) are calculated. The first is sum of elements from the row (the number of total paths), the second is the weighted sum of elements from the row (the length of total paths).

For the graph G from figure 1 we obtain a set of ten tables with three different ordering criteria (table 1-10).

Table 1. APS and LAPS numbered sequences

APS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$	LAPS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$
1	2	3	4	7	6	2	1	25	97	$1=\sigma(8)$	1	2	3	4	5	5	2	22	99
2	2	3	5	5	6	4	0	25	97	$2=\sigma(1)$	2	3	4	7	6	2	1	25	97
3	3	3	5	7	3	0	0	21	67	$3=\sigma(2)$	2	3	5	5	6	4	0	25	97
4	3	4	4	6	3	1	0	21	68	$4=\sigma(6)$	2	4	4	5	6	3	0	24	90
5	2	4	5	5	4	5	1	26	102	$5=\sigma(5)$	2	4	5	5	4	5	1	26	102
6	2	4	4	5	6	3	0	24	90	$6=\sigma(7)$	3	3	4	5	5	2	0	22	78
7	3	3	4	5	5	2	0	22	78	$7=\sigma(3)$	3	3	5	7	3	0	0	21	67
8	1	2	3	4	5	5	2	22	99	$8=\sigma(4)$	3	4	4	6	3	1	0	21	68

Table 2. DPS and LDPS numbered sequences

DPS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$	LDPS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$
1	2	3	2	2	0	0	0	9	22	$1=\sigma(8)$	1	2	3	2	0	0	0	8	22
2	2	3	3	0	0	0	0	8	17	$2=\sigma(1)$	2	3	2	2	0	0	0	9	22
3	3	3	1	0	0	0	0	7	12	$3=\sigma(2)$	2	3	3	0	0	0	0	8	17
4	3	4	0	0	0	0	0	7	11	$4=\sigma(5)$	2	4	1	0	0	0	0	7	13
5	2	4	1	0	0	0	0	7	13	$5=\sigma(6)$	2	4	2	0	0	0	0	8	16
6	2	4	2	0	0	0	0	8	16	$6=\sigma(3)$	3	3	1	0	0	0	0	7	12
7	3	3	2	0	0	0	0	8	15	$7=\sigma(7)$	3	3	2	0	0	0	0	8	15
8	1	2	3	2	0	0	0	8	22	$8=\sigma(4)$	3	4	0	0	0	0	0	7	11

Table 3. Δ PS and Δ LAPS numbered sequences

Δ PS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$	Δ LAPS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$
1	0	0	0	1	4	2	1	8	43	$1=\sigma(1)$	0	0	0	1	4	2	1	8	43
2	0	0	0	1	4	4	0	9	48	$2=\sigma(6)$	0	0	0	1	4	3	0	8	42
3	0	0	2	4	3	0	0	9	37	$3=\sigma(2)$	0	0	0	1	4	4	0	9	48
4	0	0	2	4	2	1	0	9	38	$4=\sigma(5)$	0	0	0	4	0	4	1	9	47
5	0	0	0	4	0	4	1	9	47	$5=\sigma(4)$	0	0	2	4	2	1	0	9	38
6	0	0	0	1	4	3	0	8	42	$6=\sigma(3)$	0	0	2	4	3	0	0	9	37
7	1	0	0	1	4	2	0	8	37	$7=\sigma(8)$	1	0	0	0	1	4	2	8	44
8	1	0	0	0	1	4	2	8	44	$8=\sigma(7)$	1	0	0	1	4	2	0	8	37

Table 4. ATS and LATS numbered sequences

ATS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$	LATS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$
1	0	0	0	3	4	1	1	9	45	$1=\sigma(8)$	0	0	0	0	1	3	2	6	37
2	0	0	1	1	3	4	0	9	46	$2=\sigma(1)$	0	0	0	3	4	1	1	9	45
3	0	0	1	5	3	0	0	9	38	$3=\sigma(5)$	0	0	1	1	0	4	1	7	38
4	0	1	0	3	2	1	0	7	30	$4=\sigma(2)$	0	0	1	1	3	4	0	9	46
5	0	0	1	1	0	4	1	7	38	$5=\sigma(3)$	0	0	1	5	3	0	0	9	38
6	0	1	0	0	3	3	0	7	35	$6=\sigma(6)$	0	1	0	0	3	3	0	7	35
7	1	0	0	1	3	2	0	7	32	$7=\sigma(4)$	0	1	0	3	2	1	0	7	30
8	0	0	0	0	1	3	2	6	37	$8=\sigma(7)$	1	0	0	1	3	2	0	7	32

Table 5. DTS and LDTS numbered sequences

DTS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$	LDTS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$
1	0	0	0	2	0	0	0	2	8	$1=\sigma(8)$	0	0	0	0	0	0	0	0	0
2	0	0	1	0	0	0	0	1	3	$2=\sigma(1)$	0	0	0	2	0	0	0	2	8
3	0	0	1	0	0	0	0	1	3	$3=\sigma(3)$	0	0	1	0	0	0	0	1	3
4	0	1	0	0	0	0	0	1	2	$4=\sigma(5)$	0	0	1	0	0	0	0	1	3
5	0	0	1	0	0	0	0	1	3	$5=\sigma(2)$	0	0	1	0	0	0	0	1	3
6	0	1	0	0	0	0	0	1	2	$6=\sigma(4)$	0	1	0	0	0	0	0	1	2
7	1	0	0	0	0	0	0	1	1	$7=\sigma(6)$	0	1	0	0	0	0	0	1	2
8	0	0	0	0	0	0	0	0	0	$8=\sigma(7)$	1	0	0	0	0	0	0	1	1

Table 6. Δ TS and Δ LATS numbered sequences

Δ TS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$	Δ LATS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$
1	0	0	0	0	2	1	1	4	23	$1=\sigma(8)$	0	0	0	0	0	2	2	4	26
2	0	0	0	0	1	4	0	5	29	$2=\sigma(5)$	0	0	0	0	0	3	1	4	25
3	0	0	0	2	3	0	0	5	23	$3=\sigma(6)$	0	0	0	0	1	3	0	4	23
4	0	0	0	2	1	1	0	4	19	$4=\sigma(2)$	0	0	0	0	1	4	0	5	29
5	0	0	0	0	0	3	1	4	25	$5=\sigma(1)$	0	0	0	0	2	1	1	4	23
6	0	0	0	0	1	3	0	4	23	$6=\sigma(4)$	0	0	0	2	1	1	0	4	19
7	1	0	0	0	2	2	0	5	23	$7=\sigma(3)$	0	0	0	2	3	0	0	5	23
8	0	0	0	0	0	2	2	4	26	$8=\sigma(7)$	1	0	0	0	2	2	0	5	23

Table 7. AVS and LAVS numbered sequences

AVS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$	LAVS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$
1	3	6	7	8	8	8	8	48	212	$1=\sigma(8)$	2	4	7	8	8	8	8	45	207
2	3	6	8	8	8	8	8	49	215	$2=\sigma(1)$	3	6	7	8	8	8	8	48	212
3	4	7	8	8	8	8	8	51	218	$3=\sigma(2)$	3	6	8	8	8	8	8	49	215
4	4	8	8	8	8	8	8	52	220	$4=\sigma(5)$	3	7	8	8	8	8	8	50	217
5	3	7	8	8	8	8	8	50	217	$5=\sigma(6)$	3	7	8	8	8	8	8	50	217
6	3	7	8	8	8	8	8	50	217	$6=\sigma(3)$	4	7	8	8	8	8	8	51	218
7	4	7	8	8	8	8	8	51	218	$7=\sigma(7)$	4	7	8	8	8	8	8	51	218
8	2	4	7	8	8	8	8	45	207	$8=\sigma(4)$	4	8	8	8	8	8	8	52	220

Table 8. DVS and LDVS numbered sequences

DVS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$	LDVS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$
1	2	3	1	1	0	0	0	7	15	$1=\sigma(8)$	1	2	3	1	0	0	0	7	18
2	2	3	2	0	0	0	0	7	14	$2=\sigma(1)$	2	3	1	1	0	0	0	7	15
3	3	3	1	0	0	0	0	7	12	$3=\sigma(2)$	2	3	2	0	0	0	0	7	14
4	3	4	0	0	0	0	0	7	11	$4=\sigma(5)$	2	4	1	0	0	0	0	7	13
5	2	4	1	0	0	0	0	7	13	$5=\sigma(6)$	2	4	1	0	0	0	0	7	13
6	2	4	1	0	0	0	0	7	13	$6=\sigma(3)$	3	3	1	0	0	0	0	7	12
7	3	3	1	0	0	0	0	7	12	$7=\sigma(7)$	3	3	1	0	0	0	0	7	12
8	1	2	3	1	0	0	0	7	18	$8=\sigma(4)$	3	4	0	0	0	0	0	7	11

Table 9. Δ VS and Δ LVS numbered sequences

Δ VS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$	Δ LVS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$
1	0	0	0	1	3	2	1	7	38	$1=\sigma(1)$	0	0	0	1	3	2	1	7	38
2	0	0	0	1	3	3	0	7	37	$2=\sigma(2)$	0	0	0	1	3	3	0	7	37
3	0	0	1	3	3	0	0	7	30	$3=\sigma(6)$	0	0	0	1	3	3	0	7	37
4	0	0	1	3	2	1	0	7	31	$4=\sigma(5)$	0	0	0	2	0	4	1	7	39
5	0	0	0	2	0	4	1	7	39	$5=\sigma(4)$	0	0	1	3	2	1	0	7	31
6	0	0	0	1	3	3	0	7	37	$6=\sigma(3)$	0	0	1	3	3	0	0	7	30
7	1	0	0	1	3	2	0	7	32	$7=\sigma(8)$	1	0	0	0	1	3	2	7	38
8	1	0	0	0	1	3	2	7	38	$8=\sigma(7)$	1	0	0	1	3	2	0	7	32

Table 10. ACS and LACS numbered sequences

ACS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$	LACS	1	2	3	4	5	6	7	$\Sigma_i S_{i,j}$	$\Sigma_i S_{i,j} \cdot i$
1	0	0	0	1	1	0	0	2	11	$1=\sigma(8)$	0	0	0	0	0	0	0	0	0
2	0	0	0	1	1	0	0	2	11	$2=\sigma(1)$	0	0	0	0	1	1	0	2	11
3	0	0	0	2	1	0	0	3	16	$3=\sigma(2)$	0	0	0	0	1	1	0	2	11
4	0	0	0	2	1	0	0	3	16	$4=\sigma(6)$	0	0	0	0	1	1	0	2	11
5	0	0	0	2	0	0	0	2	10	$5=\sigma(7)$	0	0	0	0	1	1	0	2	11
6	0	0	0	1	1	0	0	2	11	$6=\sigma(5)$	0	0	0	0	2	0	0	2	10
7	0	0	0	1	1	0	0	2	11	$7=\sigma(3)$	0	0	0	0	2	1	0	3	16
8	0	0	0	0	0	0	0	0	0	$8=\sigma(4)$	0	0	0	0	2	1	0	3	16

4. Graph Partitioning

Three operators on SMN matrices were applied for vertex comparison: LSMN (lexicographically ordered matrix of numbering), $SMN\Sigma_p n_p$ (sum of all path number that satisfy selected criteria) and $SMN\Sigma_p n_p \cdot l(p)$ (sum of all length of path that satisfy selected criteria).

Every operator induces a partial ordering in V (the set of vertices form the graph), and partitions the vertices in classes of equivalence that it correspond to equal operator values.

For example, 1 is equivalent with 2 by $AP\Sigma_p n_p$, with 6,7 and 8 by $\Delta P\Sigma_p n_p$, with 2, 6, 7 by ACS and $ACS\Sigma_p n_p \cdot l(p)$, with 2, 5, 6, 7 by $ACS\Sigma_p n_p$, with 4, 5, 6 and 8 by $\Delta T\Sigma_p n_p$. All vertices are equivalent by $\Delta V\Sigma_p n_p$ and $DV\Sigma_p n_p$. Vertex 1 is singularly according to $AP\Sigma_p n_p \cdot l(p)$, $DP\Sigma_p n_p$, $\Delta P\Sigma_p n_p \cdot l(p)$, $AT\Sigma_p n_p \cdot l(p)$, LDTS, $DT\Sigma_p n_p$, $DT\Sigma_p n_p \cdot l(p)$, LAVS, $AV\Sigma_p n_p$, $AV\Sigma_p n_p \cdot l(p)$, LDVS, $DV\Sigma_p n_p \cdot l(p)$ and $L\Delta V S$.

This example demonstrates the flexibility of sequence matrix criteria in the characterization of various properties of vertices.

Let us look at all partial orderings induced in the graph by every sequence matrix.

LAPS discriminates all vertices for the graph in fig. 1. The values obtained by applying $\Sigma_p n_p$ and $\Sigma_p n_p \cdot l(p)$ operators are degenerate in the case of APS. $AP\Sigma_p n_p$ partitioning $V(G)$ in 5 classes of equivalence:

$$P(V(G), AP\Sigma_p n_p) = \{(3,4), (7,8), (6), (1,2), (5)\}.$$

$AP\Sigma_p n_p \cdot l(p)$ partitions $V(G)$ in 7 classes of equivalence:

$$P(V(G), APS_{\Sigma_p n_p} \cdot l(p)) = \{(7), (8), (6), (4), (2,3), (1), (5)\}.$$

L DPS discriminates all vertices for the graph in fig. 1. The values for $\Sigma_p n_p$ and $\Sigma_p n_p \cdot l(p)$ operators are degenerate in the case of DPS.

DPS $\Sigma_p n_p$ partitions $V(G)$ in 3 classes of equivalence:

$$P(V(G), DPS_{\Sigma_p n_p}) = \{(3,4,5), (2,6,7,8), (1)\}.$$

DPS $\Sigma_p n_p \cdot l(p)$ partitions $V(G)$ in 7 classes of equivalence:

$$P(V(G), DPS_{\Sigma_p n_p} \cdot l(p)) = \{(8), (6), (4), (7), (5), (3), (1,2)\}.$$

All vertices for the graph from fig. 1 are discriminated by L Δ PS criterion. The values for $\Sigma_p n_p$ and $\Sigma_p n_p \cdot l(p)$ operators are degenerate according to Δ PS criterion.

Δ PS $\Sigma_p n_p$ partitions $V(G)$ in 2 classes of equivalence:

$$P(V(G), \Delta PS_{\Sigma_p n_p}) = \{(1,6,7,8), (2,3,4,5)\}.$$

Δ PS $\Sigma_p n_p \cdot l(p)$ partitions $V(G)$ in 7 classes of equivalence:

$$P(V(G), \Delta PS_{\Sigma_p n_p} \cdot l(p)) = \{(3,7), (4), (6), (1), (8), (5), (2)\}.$$

LATS discriminates all vertices for the graph G . The values for $\Sigma_p n_p$ and $\Sigma_p n_p \cdot l(p)$ are degenerate according to ATS.

ATS $\Sigma_p n_p$ partitions $V(G)$ in 3 classes of equivalence:

$$P(V(G), ATS_{\Sigma_p n_p}) = \{(8), (4,5,6,7), (1,2,3)\}.$$

ATS $\Sigma_p n_p \cdot l(p)$ partitions $V(G)$ in 7 classes of equivalence:

$$P(V(G), ATS_{\Sigma_p n_p} \cdot l(p)) = \{(4), (7), (6), (8), (3,5), (1), (2)\}.$$

LDTS partitions the vertices of G in 5 classes of equivalence:

$$P(V(G), LDTS) = \{(8), (1), (2,3,5), (4,6), (7)\}.$$

The values for $\Sigma_p n_p$ and $\Sigma_p n_p \cdot l(p)$ operators are degenerate according to DTS.

DTS $\Sigma_p n_p$ partitions $V(G)$ in 3 classes of equivalence:

$$P(V(G), DTS_{\Sigma_p n_p}) = \{(8), (2,3,4,5,6,7), (1)\}.$$

DTS $\Sigma_p n_p \cdot l(p)$ partitions $V(G)$ in 5 classes of equivalence:

$$P(V(G), DTS_{\Sigma_p n_p} \cdot l(p)) = \{(8), (7), (4,6), (2,3,5), (1)\}.$$

L Δ TS discriminates all vertices for the graph in fig. 1. The values for $\Sigma_p n_p$ and $\Sigma_p n_p \cdot l(p)$ operators are degenerate in Δ TS criteria.

Δ TS $\Sigma_p n_p$ partitions $V(G)$ in 2 classes of equivalence:

$$P(V(G), \Delta TS_{\Sigma_p n_p}) = \{(1,4,5,6,8), (2,3,7)\}.$$

Δ TS $\Sigma_p n_p \cdot l(p)$ partitions $V(G)$ in 5 classes of equivalence:

$$P(V(G), \Delta TS_{\Sigma_p n_p} \cdot l(p)) = \{(4), (1,3,6,7), (5), (8), (2)\}.$$

LAVS partitions the vertices of G in 6 classes of equivalence:

$$P(V(G), LAVS) = \{(8), (1), (2), (5,6), (3,7), (4)\}.$$

The values for Σ_{pn_p} and $\Sigma_{pn_p \cdot l(p)}$ are degenerate in AVS criteria.

$AVS_{\Sigma_{pn_p}}$ partitions $V(G)$ in 6 classes of equivalence:

$$P(V(G), AVS_{\Sigma_{pn_p}}) = \{(8), (1), (2), (5,6), (3,7), (4)\}.$$

$AVS_{\Sigma_{pn_p \cdot l(p)}}$ partitions $V(G)$ in 6 classes of equivalence:

$$P(V(G), AVS_{\Sigma_{pn_p \cdot l(p)}}) = \{(8), (1), (2), (5,6), (3,7), (4)\}.$$

LDVS partitions the vertices of G in 6 classes of equivalence:

$$P(V(G), LDVS) = \{(8), (1), (2), (5,6), (3,7), (4)\}.$$

The values for Σ_{pn_p} and $\Sigma_{pn_p \cdot l(p)}$ are degenerate in DVS criteria.

$DVS_{\Sigma_{pn_p}}$ does not discriminate the vertices of G :

$$P(V(G), DVS_{\Sigma_{pn_p}}) = \{(1,2,3,4,5,6,7,8)\}.$$

$DVS_{\Sigma_{pn_p \cdot l(p)}}$ partitions $V(G)$ in 6 classes of equivalence:

$$P(V(G), DVS_{\Sigma_{pn_p \cdot l(p)}}) = \{(4), (3,7), (5,6), (2), (1), (8)\}.$$

ΔVS partitions $V(G)$ in 7 classes of equivalence:

$$P(V(G), \Delta VS) = \{(1), (2,6), (5), (4), (3), (8), (7)\}.$$

The values for Σ_{pn_p} and $\Sigma_{pn_p \cdot l(p)}$ operators are totally degenerate in ΔVS criteria.

$\Delta VS_{\Sigma_{pn_p}}$ partitions does not discriminate the vertices:

$$P(V(G), \Delta VS_{\Sigma_{pn_p}}) = \{(1,2,3,4,5,6,7,8)\}.$$

$\Delta VS_{\Sigma_{pn_p \cdot l(p)}}$ partitions $V(G)$ in 6 classes of equivalence:

$$P(V(G), \Delta VS_{\Sigma_{pn_p \cdot l(p)}}) = \{(3), (4), (7), (2,6), (1,8), (5)\}.$$

LACS partitions $V(G)$ in 4 classes of equivalence:

$$P(V(G), LACS) = \{(8), (1,2,6,7), (5), (3,4)\}.$$

The values for Σ_{pn_p} and $\Sigma_{pn_p \cdot l(p)}$ are degenerate in ACS.

$ACS_{\Sigma_{pn_p}}$ partitions $V(G)$ in 3 classes:

$$P(V(G), ACS_{\Sigma_{pn_p}}) = \{(8), (1,2,5,6,7), (3,4)\}.$$

$ACS_{\Sigma_{pn_p \cdot l(p)}}$ partitions $V(G)$ in 4 classes of equivalence:

$$P(V(G), ACS_{\Sigma_{pn_p \cdot l(p)}}) = \{(8), (5), (1,2,6,7), (3,4)\}.$$

Note that the matrices LAVS and LDVS contain different values, but induce the same ordering " σ " for vertices:

$$\sigma_{LAVS} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 6 & 8 & 4 & 5 & 7 & 1 \end{pmatrix} = \sigma_{LDVS} \quad (\sigma P)$$

The property (σ P) was verified by a set of various graphs, with different numbers of cycles, edges and vertices. In each case, the LAVS and LDVS criteria induce the same partial ordering for vertices.

5. Vertices Coloring

According to the partitions induced and presented in tables 1-10 a graph vertices coloring are making. In colorings, a code color was used. In increasing order of values for selected criteria, the colors used are red, gold, blue, lavender, lime, green and brown. For the partitions that discriminate all vertices no coloring was made.

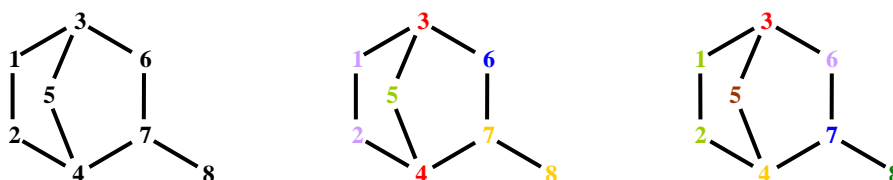


Fig. 3. Graph G vertices colorings from fig. 1; partitions: LAPS, $APS\Sigma_{pn_p}$ and $APS\Sigma_{pn_p}\cdot l(p)$

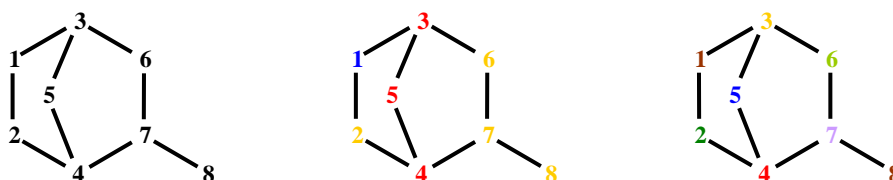


Fig. 4. Graph G vertices colorings from fig. 1; partitions: LDPS, $DPS\Sigma_{pn_p}$ and $DPS\Sigma_{pn_p}\cdot l(p)$

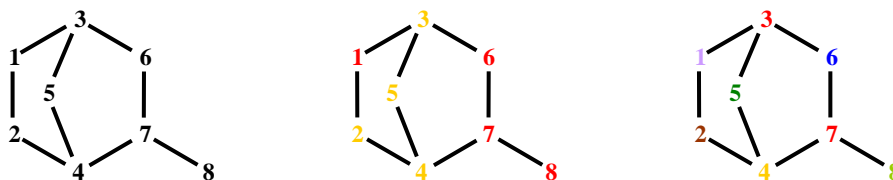


Fig. 5. Graph G vertices colorings from fig. 1; partitions: LAPPS, $\Delta PS\Sigma_{pn_p}$ and $\Delta PS\Sigma_{pn_p}\cdot l(p)$

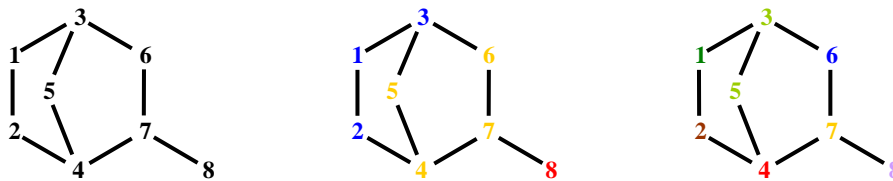


Fig. 6. Graph G vertices colorings from fig. 1; partitions: LATS, $AT\Sigma_{pn_p}$ and $AT\Sigma_{pn_p}.l(p)$

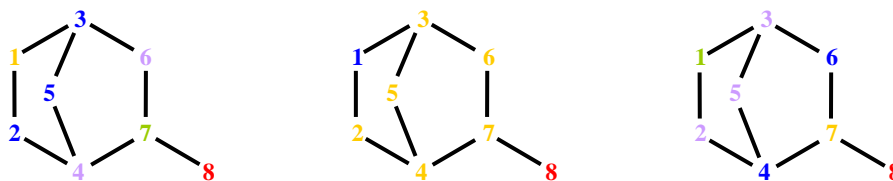


Fig. 7. Graph G vertices colorings from fig. 1; partitions: LDTS, $DT\Sigma_{pn_p}$ and $DT\Sigma_{pn_p}.l(p)$

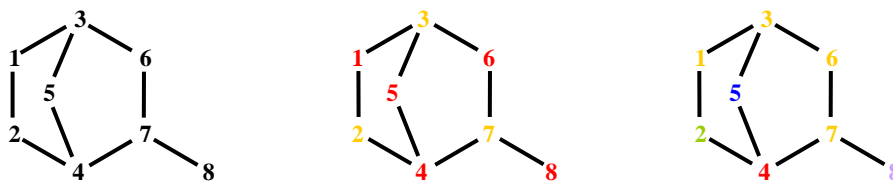


Fig. 8. Graph G vertices colorings from fig. 1; partitions: LATs, $AT\Sigma_{pn_p}$ and $AT\Sigma_{pn_p}.l(p)$

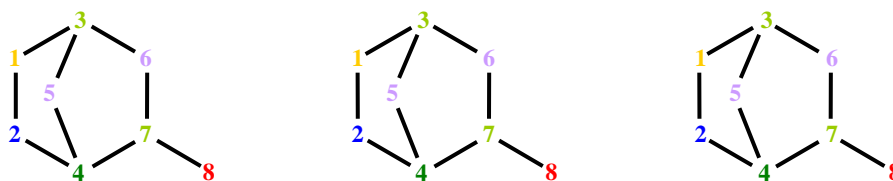


Fig. 9. Graph G vertices colorings from fig. 1; partitions: LAVS, $AV\Sigma_{pn_p}$ and $AV\Sigma_{pn_p}.l(p)$

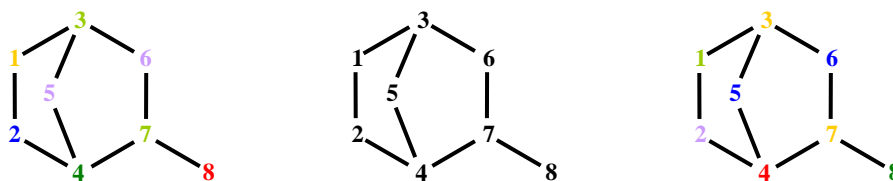


Fig. 10. Graph G vertices colorings from fig. 1; partitions: LDVS, $DV\Sigma_{pn_p}$ and $DV\Sigma_{pn_p}.l(p)$

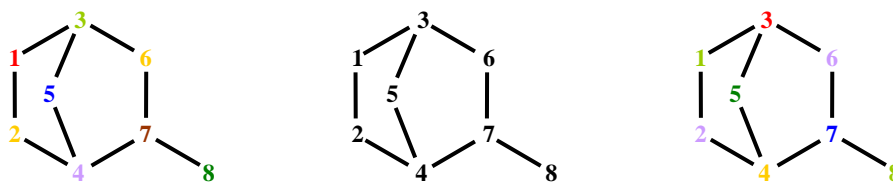


Fig. 11. Graph G vertices colorings from fig. 1; partitions: $LAVS$, $\Delta VS\Sigma_p n_p$ and $\Delta VS\Sigma_p n_p \cdot l(p)$

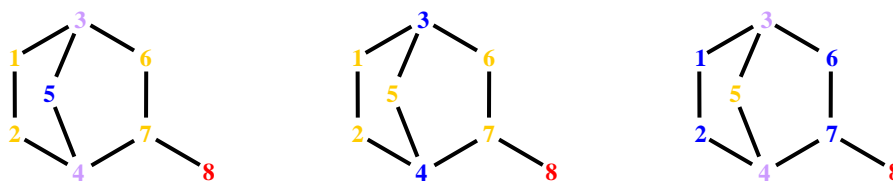


Fig. 12. Graph G vertices colorings from fig. 1; partitions: $LACS$, $ACS\Sigma_p n_p$ and $ACS\Sigma_p n_p \cdot l(p)$

6. Conclusions

The analyzed matrix critters are capable to partitioning vertices in equivalence classes by different properties (i.e. number of paths, length of paths, et all).

All criteria give a specific partitioning of vertices, excepting $LAVS$ and $LDVS$ ordering. For $LAVS$ and $LDVS$ criterion, an interesting property result: although the individual matrix values are different, comparison rules match, so that the two matrices induce identical partitioning. Testing on a large set of graphs, the partitioning rule $LADS \equiv LVDS$ remain same.

That is a subject for a new theorem that will be proved later.

More, all AVS criterion ($AVS\Sigma_p n_p$ and $AVS\Sigma_p n_p \cdot l(p)$ and $LAVS$) make same partitioning of graph vertices for the given graph G from fig. 1. More, the ordering is the same (see fig. 9).

Vertex partitioning can be applied in graph coloring [20], the theory of traffic (lights), distributed electric circuits, computer networks [21], and chemical graphs [22]. Edge partitioning is another problem that can be solved using terminal path structure. For more details about this subject, please consult [23].



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